

# A Comment on "Analysis of Video Image Sequences Using Point and Line Correspondences"

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### Abstract

In this paper we would like to deny the results of Wang et al.<sup>[1]</sup> raising two fundamental claims:

- A line does not contribute anything to recognition of motion parameters from two images
- Four traceable points are not sufficient to recover motion parameters from two perspective<sup>1</sup> projections.

To be constructive, however, we show that four traceable points are sufficient to recover motion parameters from two frames under orthogonal projection and that five points are sufficient to simplify the solution of the two-frame problem under orthogonal projection to solving a linear equation system.

**Keywords:** Structure, Motion, Point, Line, Image

## 1 Introduction

Wang<sup>[1]</sup> claimed a new method of recovering structure and motion parameters from a sequence of two frames (under perspective projection) applicable whenever four points and a straight line form a rigid body and can be traced (or, in the languages of the authors: line and points correspondence is known) from frame to frame. We show here that the claim of the Authors is wrong at least for two basic reasons:

- A line does not contribute anything to recognition of motion parameters from two images
- Four traceable points are not sufficient to recover motion parameters from two perspective projections.

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<sup>1</sup>Everywhere I replaced "prospective" with "perspective" without further notifying this

To show that a line does not contribute anything to recognition of motion parameters from two images (Section 2)

- we recall the results of Weng<sup>[2]</sup>
- we consider degrees of freedom and amount of information connected with a line in each image
- we point at errors in the proof of the results in <sup>[1]</sup>

To show, that four traceable points are not sufficient to recover motion parameters from two perspective projections (Section 3)

- we make a degrees of freedom argument
- we demonstrate that infinitely many four point objects may be constructed from two images
- we explain errors in the proof given in <sup>[1]</sup>

To complete the argument, we show that five points are sufficient for recovery of motion parameters from two perspective images (Section 4)

To be constructive, however, we show that four traceable points are sufficient to recover motion parameters from two frames under orthogonal projection. We show also that one additional traceable point or one additional frame are sufficient to simplify the solution of the structure and motion recovery problem under orthogonal projection to solving a linear equation system (Section 5).

## 2 A Line Does Not Help The Points

### 2.1 The Argument After Weng

In the paper of Weng<sup>[2]</sup> dealing with the recovery of motion and structure from line correspondences we find on page 319 the subsection "*A. Why Two Views Are Not Sufficient*". The basic argument lies in the fact that the straight line we want to recover from perspective projections must lie in every

plane containing the image of this line and the respective focal point. If we have two images, we have also two planes only to fit the recovered straight line which is almost always possible as two planes in 3-D in general intersect along a line. Though the argument of Weng<sup>[2]</sup> deals with any number of lines only, it is easily extended to the situation of Wang<sup>[1]</sup>: namely whenever points are met for two perspective images, any number of straight lines may be in general met.

## 2.2 The Degrees of Freedom Argument

We can show the uselessness of a straight line for structure recovery from two images in a still another way. If we trace a rigid body through several (e.g. two) images we have the following degrees of freedom: the sum of degrees of freedom for each element of the rigid body in the first frame and the degrees of freedom for motion for each subsequent frame (5 for orthogonal and 6 for perspective projections if no information on constraints for motion are available). On the other hand each frame contributes some information on the positions of the traced elements of the rigid body. Now we can recover the structure of the rigid body if within each but the first frame the information gained outweighs the degrees of freedom introduced by the motion so that eventually it is possible to bind all the degrees of freedom of the first frame. We can reverse this formulation: the combined information gained on each traceable element over all frames should outweigh the degrees of freedom of this element emerging in the first frame so that superfluous information summed over all traced elements binds the degrees of freedom of the motion introduced by all but first frame.

If we consider e.g. a point then we see that in the first frame it has 3 degrees of freedom (df.) (e.g. x,y,z-co-ordinates). Each frame contributes 2 independent pieces of information on a point (e.g. x,y-co-ordinates within the projection plane). Hence two projections bind 4 df. which leaves 1 superfluous df. for recovery of motion parameters.

If we consider a straight line, however, then in the first frame it has 4 degrees of freedom (df.) (e.g. x,y-co-ordinates of the crossing point of the XY-plane, the angle between the XY-plane and the straight line, and the angle of rotation around the Z axis). Each frame contributes 2 independent pieces of information on a straight line (e.g. a,b-parameters of the  $y=ax+b$  equation of the projection within the x,y-co-ordinate system of the projection plane). **Hence two projections bind 4 df. which leaves NO superfluous df. for recovery of motion parameters, hence a traced straight line does not contribute anything to reconstruction of the rigid body from two frames. Q.e.D..**

## 2.3 Errors in The Proof of Wang

Let us discuss now the contribution of Wang<sup>[1]</sup>, specifically with the proof in Section "2.3.Specifying constraints for mutually independent equations". On page 1070 it is claimed that "Once the distance and angular constraints ... are imposed, the shape of points/line configuration is uniquely determined". It is not. Any mirror-like reflection of the configuration retains the angular and distance constraints but the shape is obviously different (mirror image).

The most tricky step of the proof is step (5) (page 1069). While all the previous steps impose constraints selecting a point position out of a continuum, the step (5) simply means selection of a value out of a discrete set of (two) points. This fact should be paid a special attention when considering methods of solving the derived non-linear equation system (Section 3, page 1071). It is obvious that if the initial guess is close enough to the wrong of the two discrete values then the methods solving non-linear systems may make hardly any use of the criterion from step (5).

### 3 The Set Of Four Point Bodies Recovered From Two Prospective Projections

As we have shown in the previous section, a line is not helpful for recovering the shape and motion parameters of a rigid body from two frames under perspective projection. So either four points are sufficient for recovery or the results of Wang<sup>[1]</sup> are geometrically wrong. We claim that the latter is the case.

#### 3.1 Degrees of Freedom Argument

Let us have a closer look at degrees of freedom than in Section 2.2 of this paper. In general, with  $p$  points and  $s$  straight lines forming the rigid body traced over  $k$  frames we have the following number of degrees of freedom:

$$-1 + 3 * p + 4 * s + 6 * (k - 1)$$

The constituent -1 is due to the fact that the scaling of the object cannot be recovered under perspective projection. The factor 3 means the number of degrees of freedom for a point, 4 - for a straight line and 6 - for the motion between frames.

Now the amount of information gained within those  $k$  frames amounts to:

$$k * (2 * p + 2 * s)$$

In order to recover the structural and motion data we request that:

$$-1 + 3 * p + 4 * s + 6 * (k - 1) \leq k * (2 * p + 2 * s) \quad (1)$$

When we have to do with 2 frames ( $k=2$ ) and 4 points ( $p=4$ ,  $s=0$ ) only, we obtain:.

$$-1 + 3 * p + 4 * s + 6 * (k - 1) = -1 + 12 + 6 = 17 > k * (2 * p + 2 * s) = 2 * 2 * 4 = 16$$

which means that Wang<sup>[1]</sup> has missed the solution by exactly one degree of freedom.

Let us notice, however, that with 3 frames (k=3) and 4 points (p=4, s=0) we obtain

$$-1 + 3 * p + 4 * s + 6 * (k - 1) = -1 + 12 + 12 = 23 < k * (2 * p + 2 * s) = 3 * 2 * 4 = 24$$

ensuring the existence of a solution (see Kłopotek<sup>[3]</sup>).

Also with 2 frames (k=2) and 5 points (p=5, s=0) we obtain

$$-1 + 3 * p + 4 * s + 6 * (k - 1) = -1 + 15 + 6 = 20 = k * (2 * p + 2 * s) = 2 * 2 * 5 = 20$$

ensuring the existence of a solution (see Nagel<sup>[5]</sup>).

Also with 3 frames (k=3) and 3 points and a single line (p=3, s=1) we obtain

$$-1 + 3 * p + 4 * s + 6 * (k - 1) = -1 + 9 + 4 + 12 = 24 = k * (2 * p + 2 * s) = 3 * (6 + 2) = 24$$

ensuring the existence of a solution.

With 3 frames (k=3) and six lines (p=0, s=6) we obtain

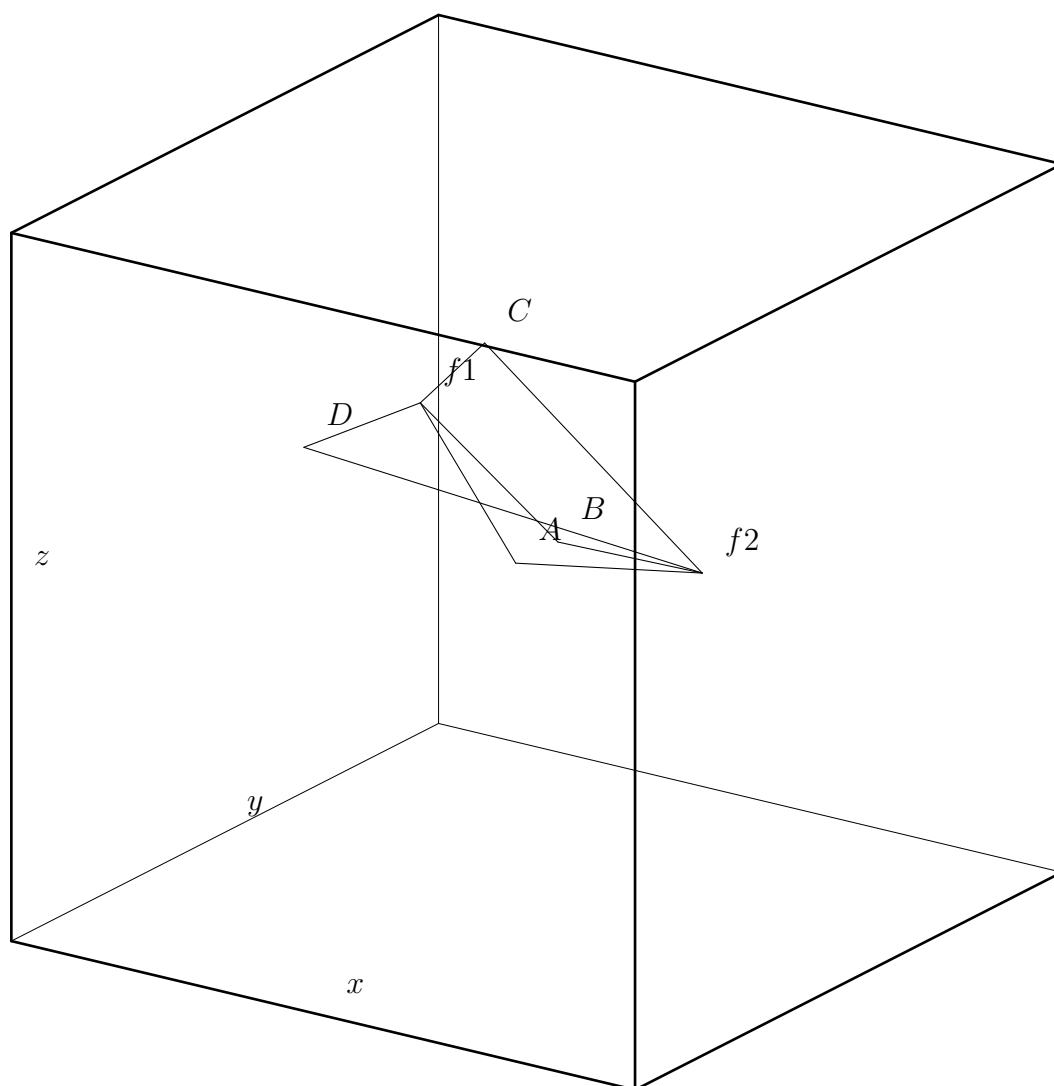
$$-1 + 3 * p + 4 * s + 6 * (k - 1) = -1 + 0 + 24 + 12 = 35 < k * (2 * p + 2 * s) = 3 * (0 + 12) = 36$$

ensuring the existence of a solution (compare Weng<sup>[2]</sup>).

The subsequent section points at the failure of Wang<sup>[1]</sup> more explicitly by demonstrating that it is possible to construct (uncountably infinitely) many objects fitting given two projection frames.

### 3.2 Objects Recoverable from Two Frames

Let us imagine now we have two perspective projections of a four point body. Instead of considering the points A',B',C',D' and A'',B'',C'',D'' being projections of the four points A,B,C,D in the first and the second projection, respectively and the focal points  $f_1$  and  $f_2$  let us consider the straight lines





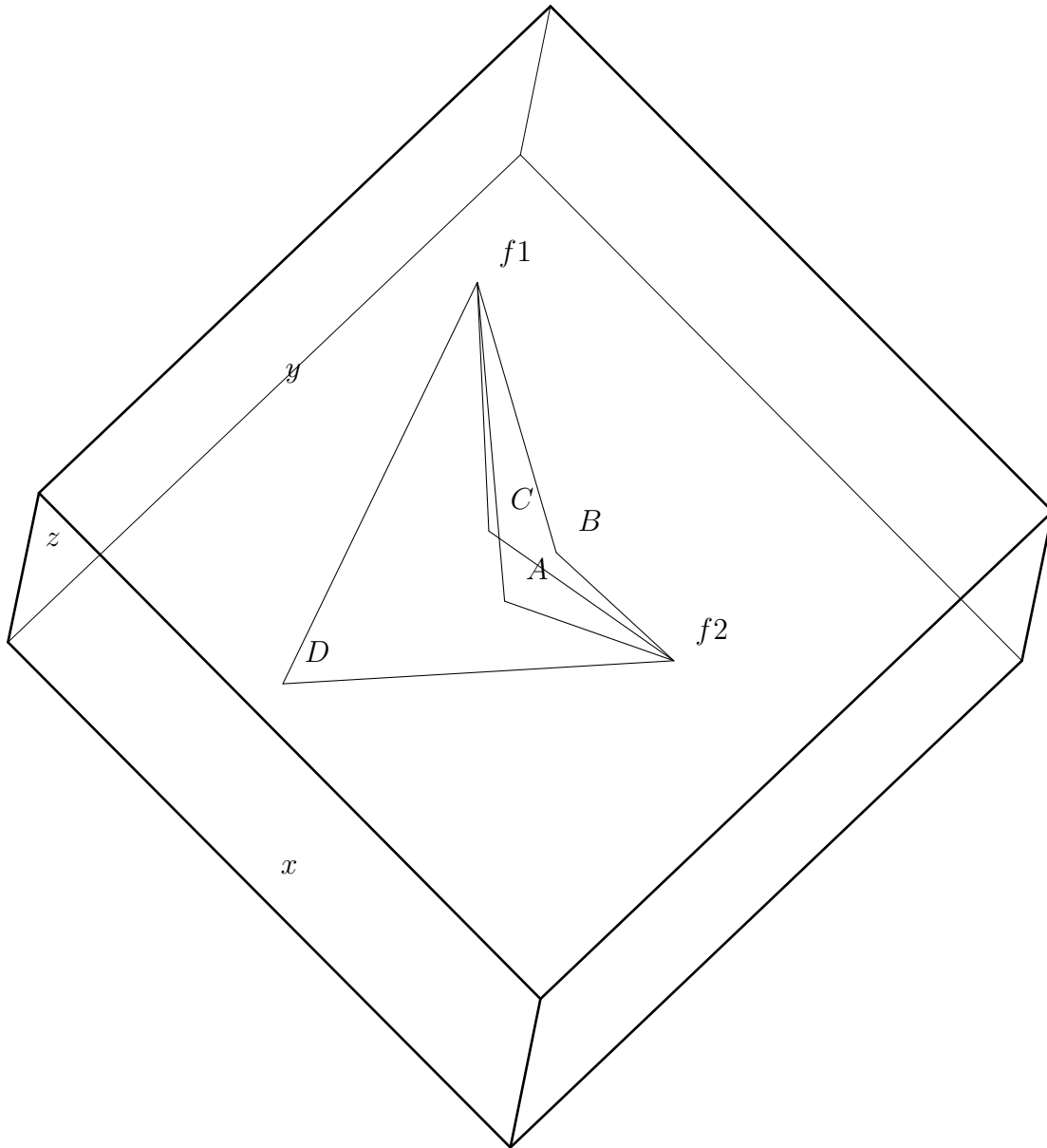
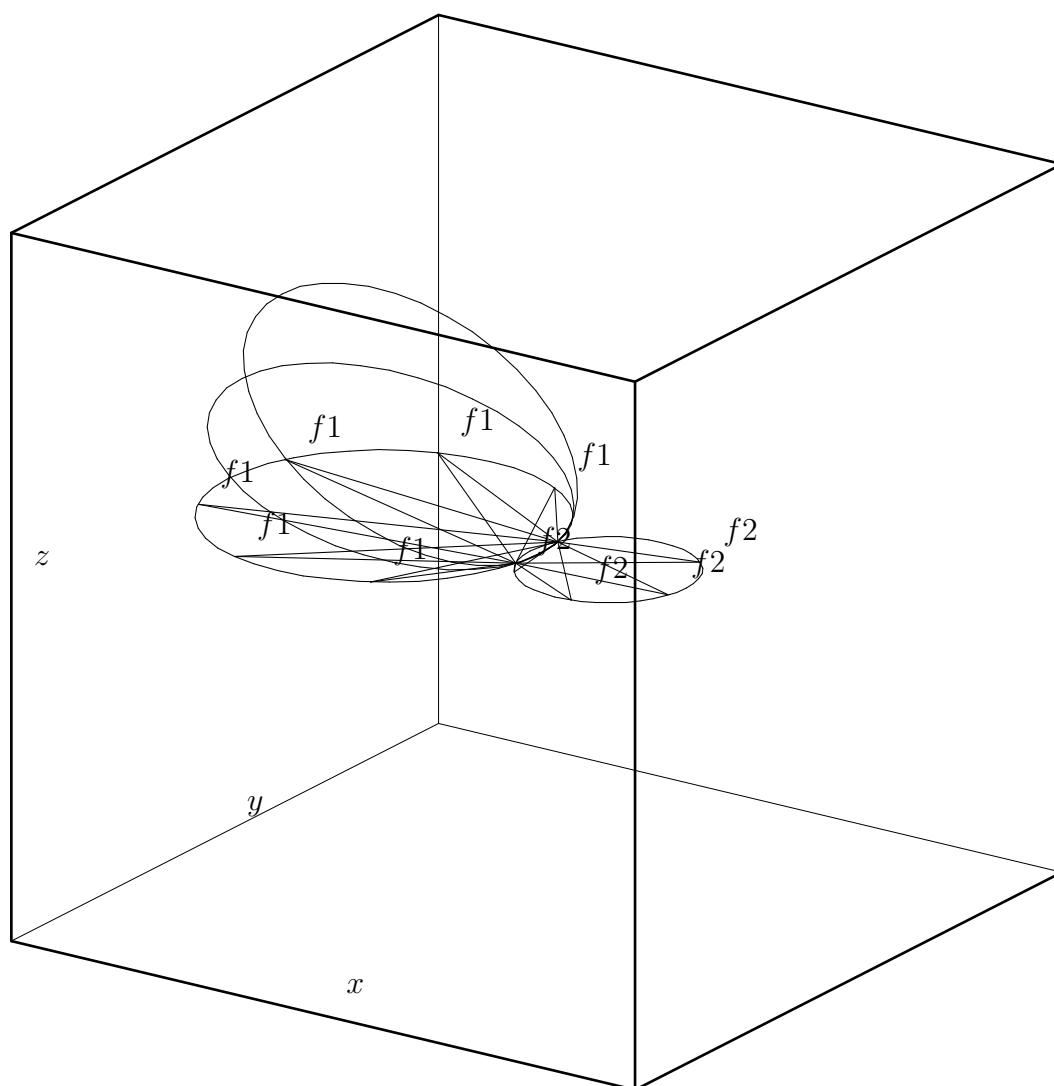


Figure 1: The basic scene, locating in space observed points  $C$  and  $D$  and focal points  $f1$  and  $f2$  from two frames assuming fixed position of observed points  $A$  and  $B$



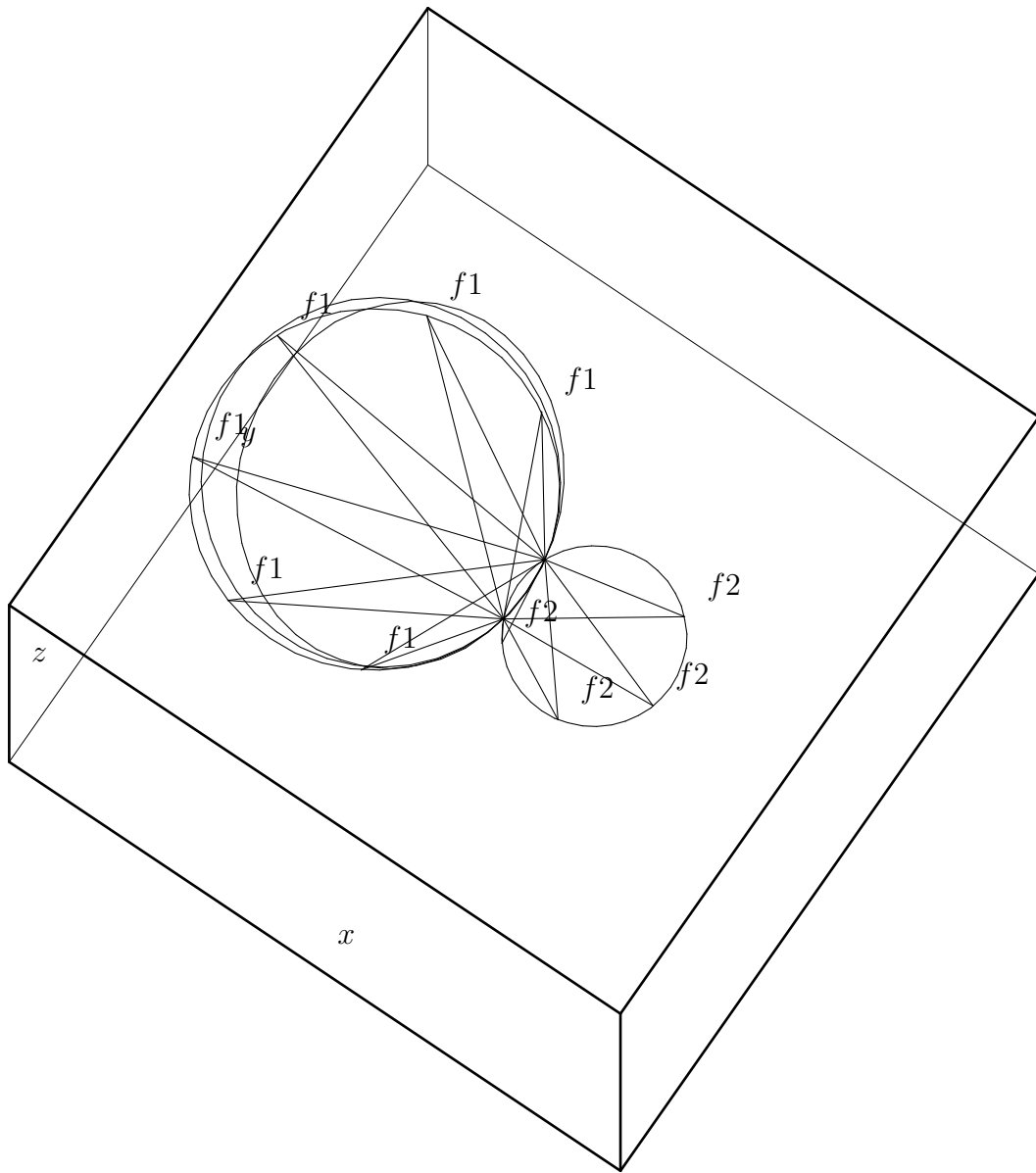
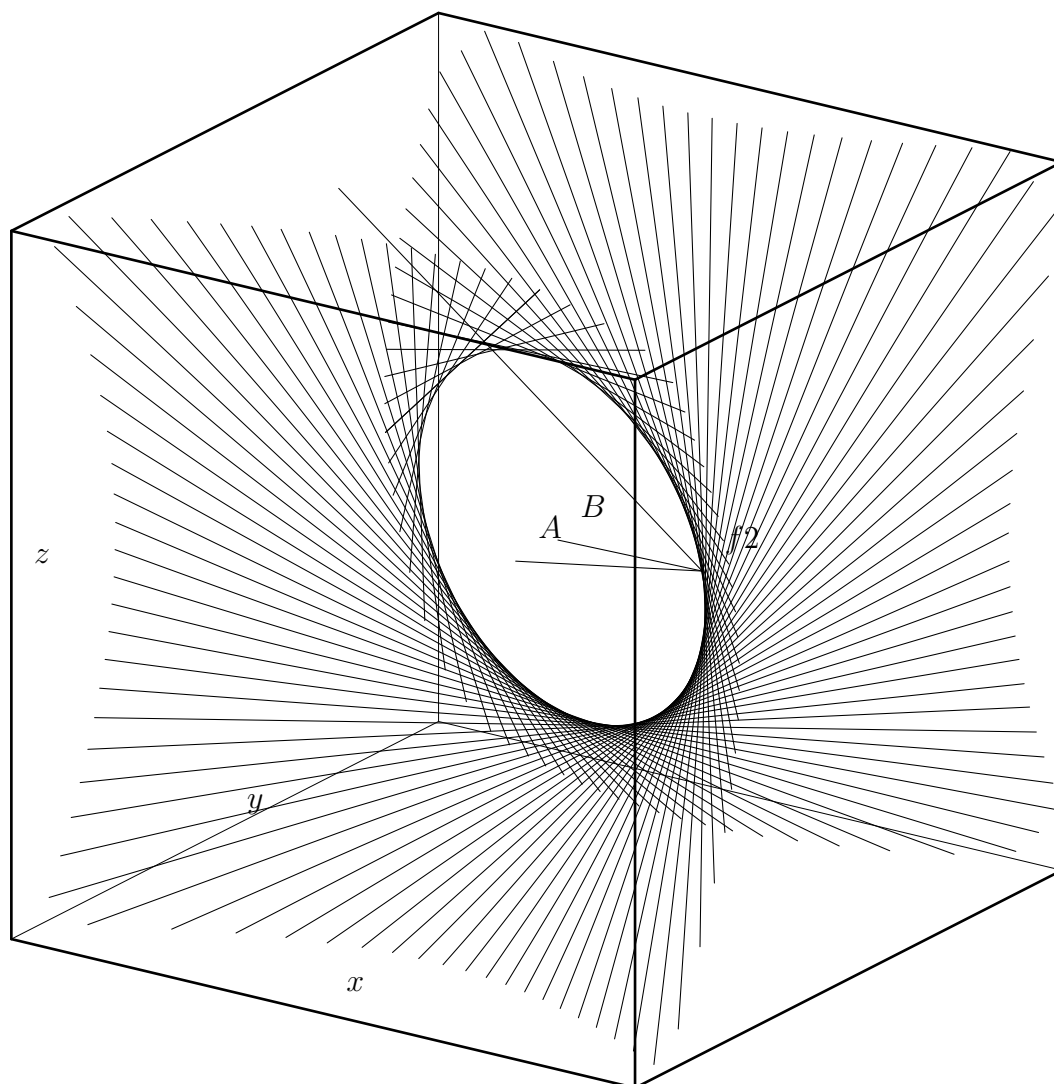


Figure 2: Degrees of freedom for locating in space focal points  $f1$  and  $f2$  from two frames assuming only two points A and B are observed



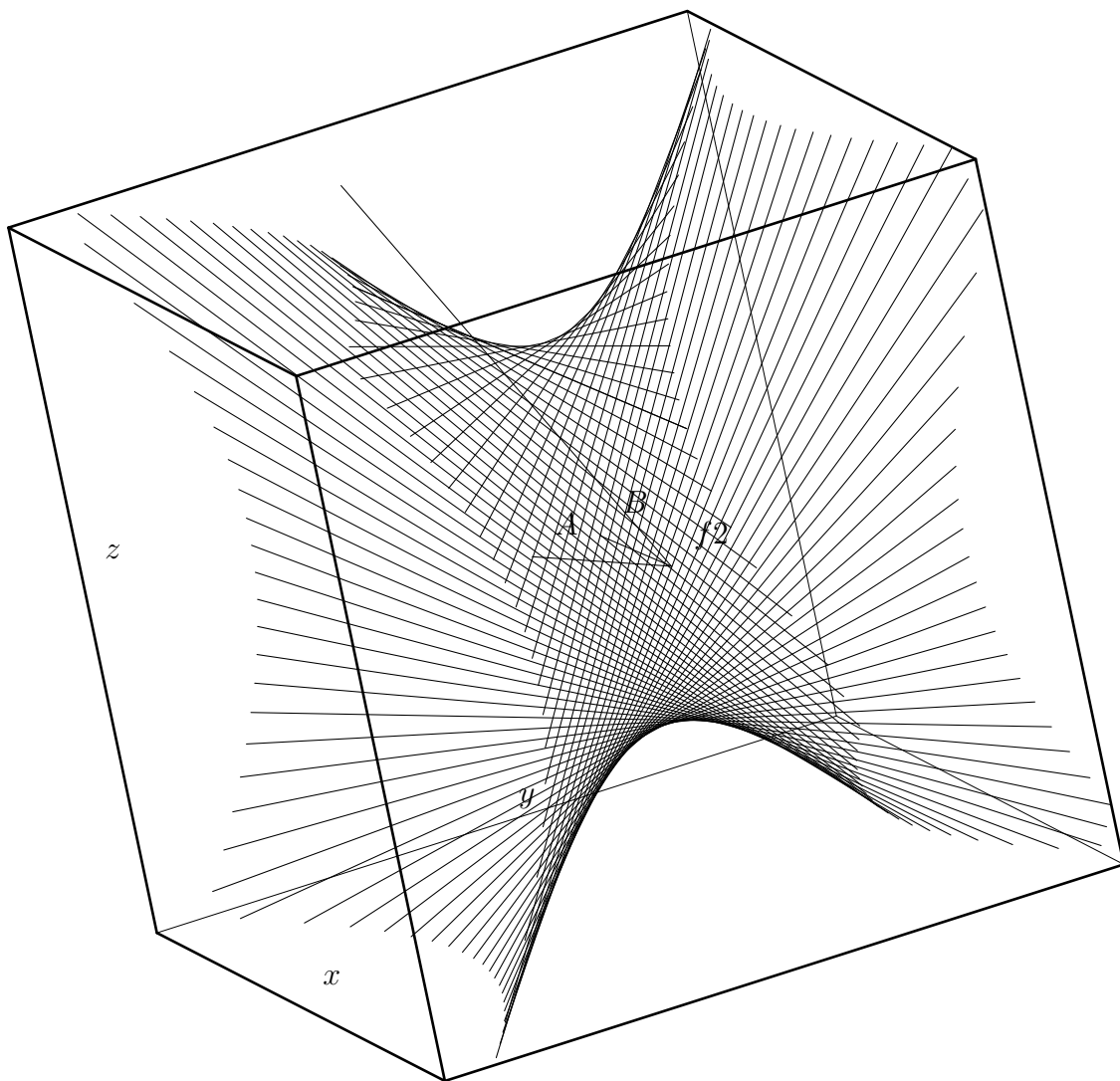
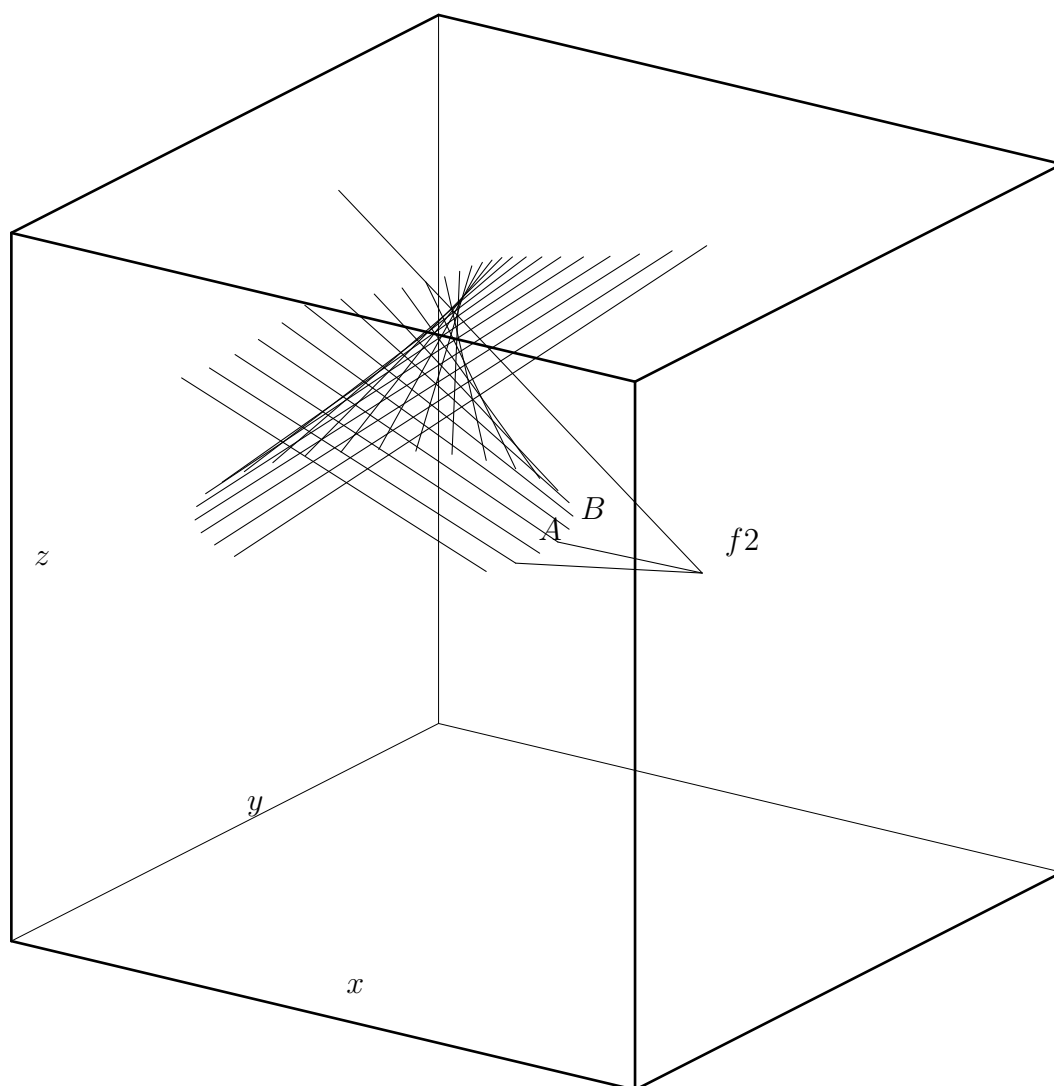


Figure 3: Three observed points: A,B,C. Assume we fix  $f1$ . Candidate surface for C when rotating  $f2$  around Y-axis.



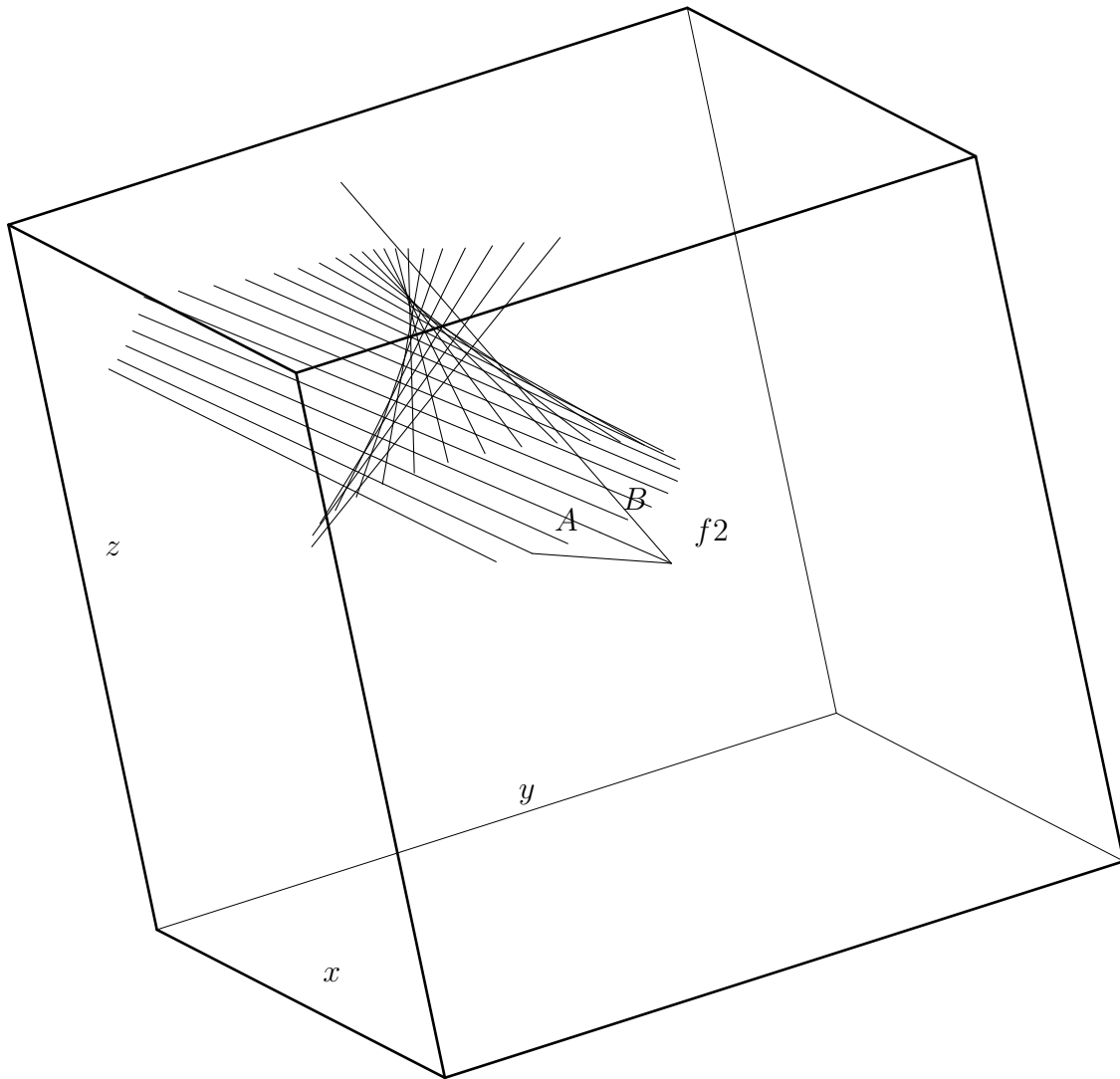
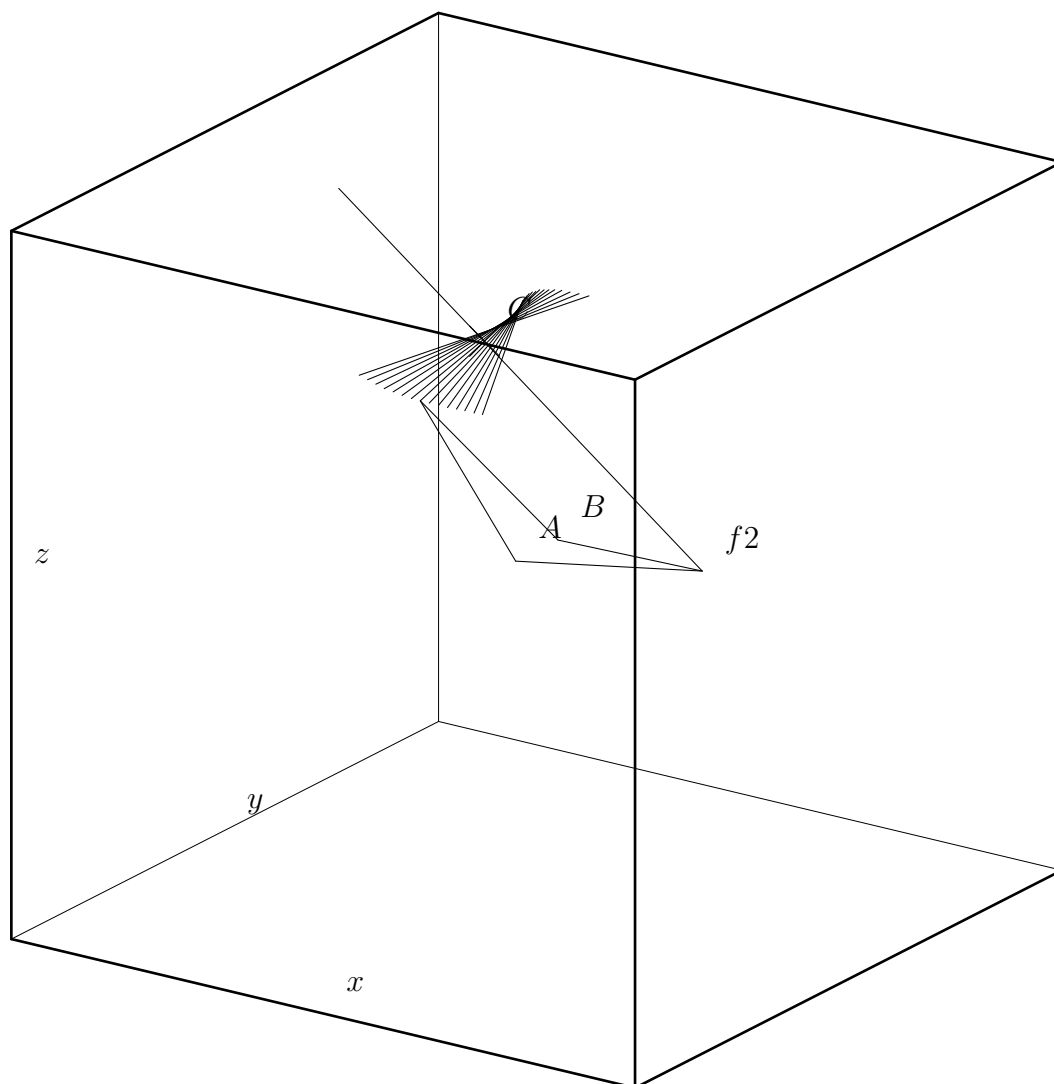


Figure 4: Three observed points: A,B,C. Assume we fix  $f1$ . Candidate surface for C when rotating  $f2$  on the  $z=0$  plane circle.





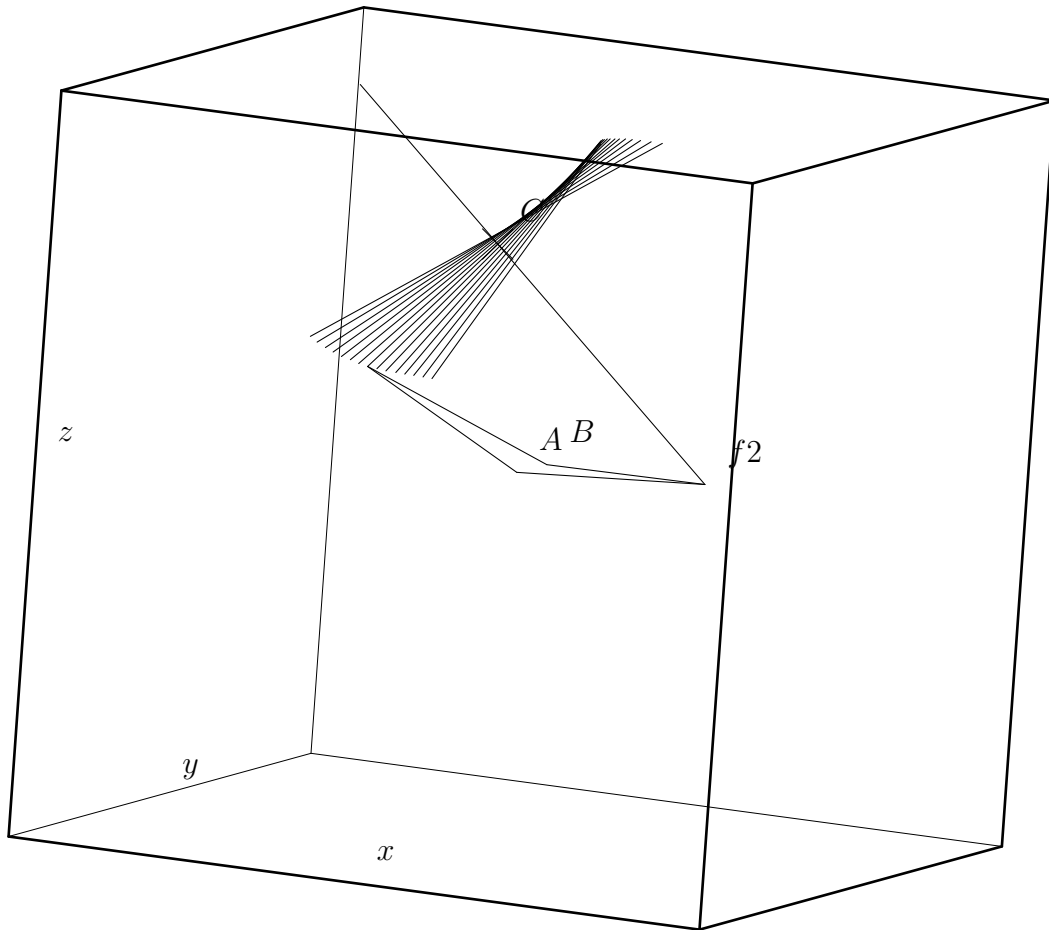
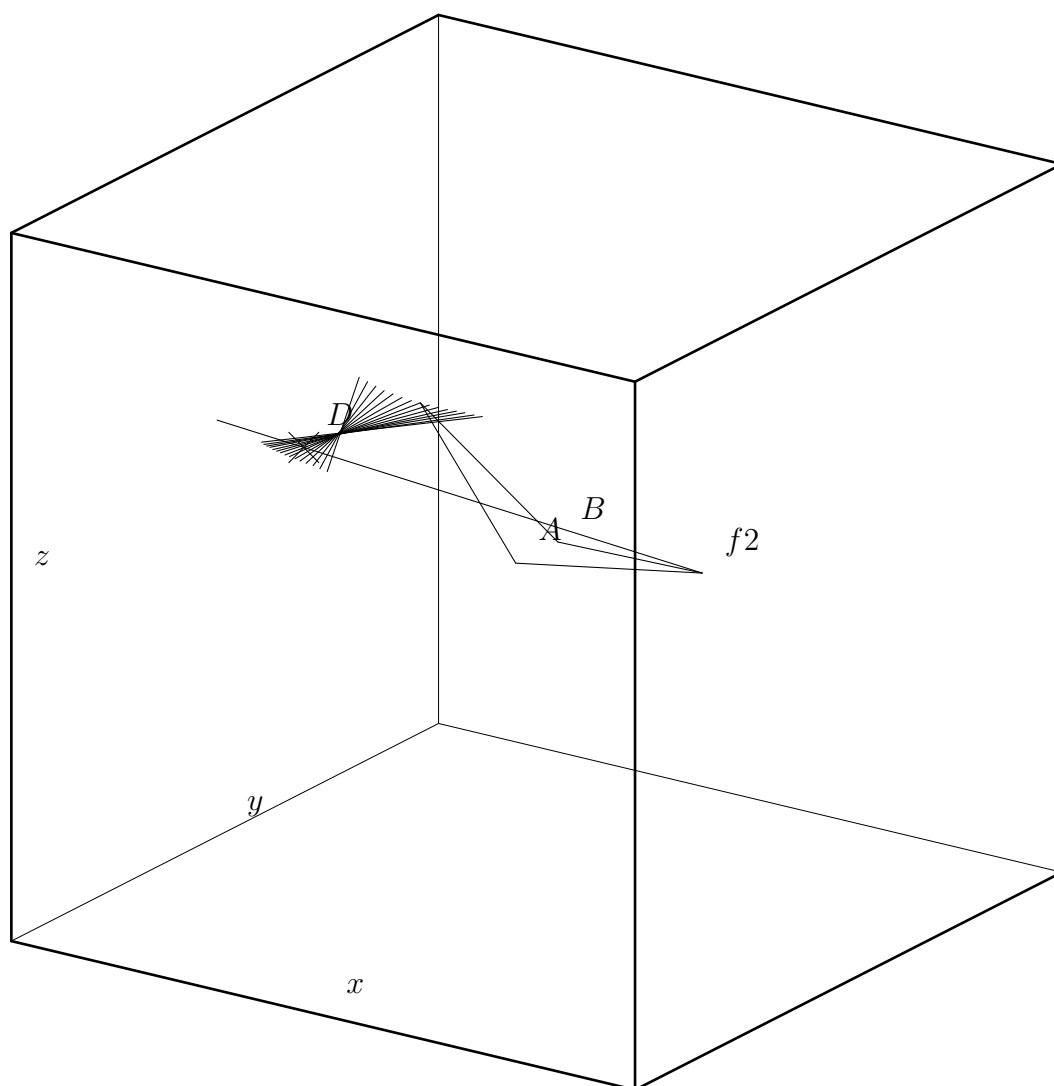


Figure 5: Three observed points: A,B,C. Assume we fix  $f_1$ . Surface consisting of lines connecting candidate C points with corresponding  $f_2$  locations.



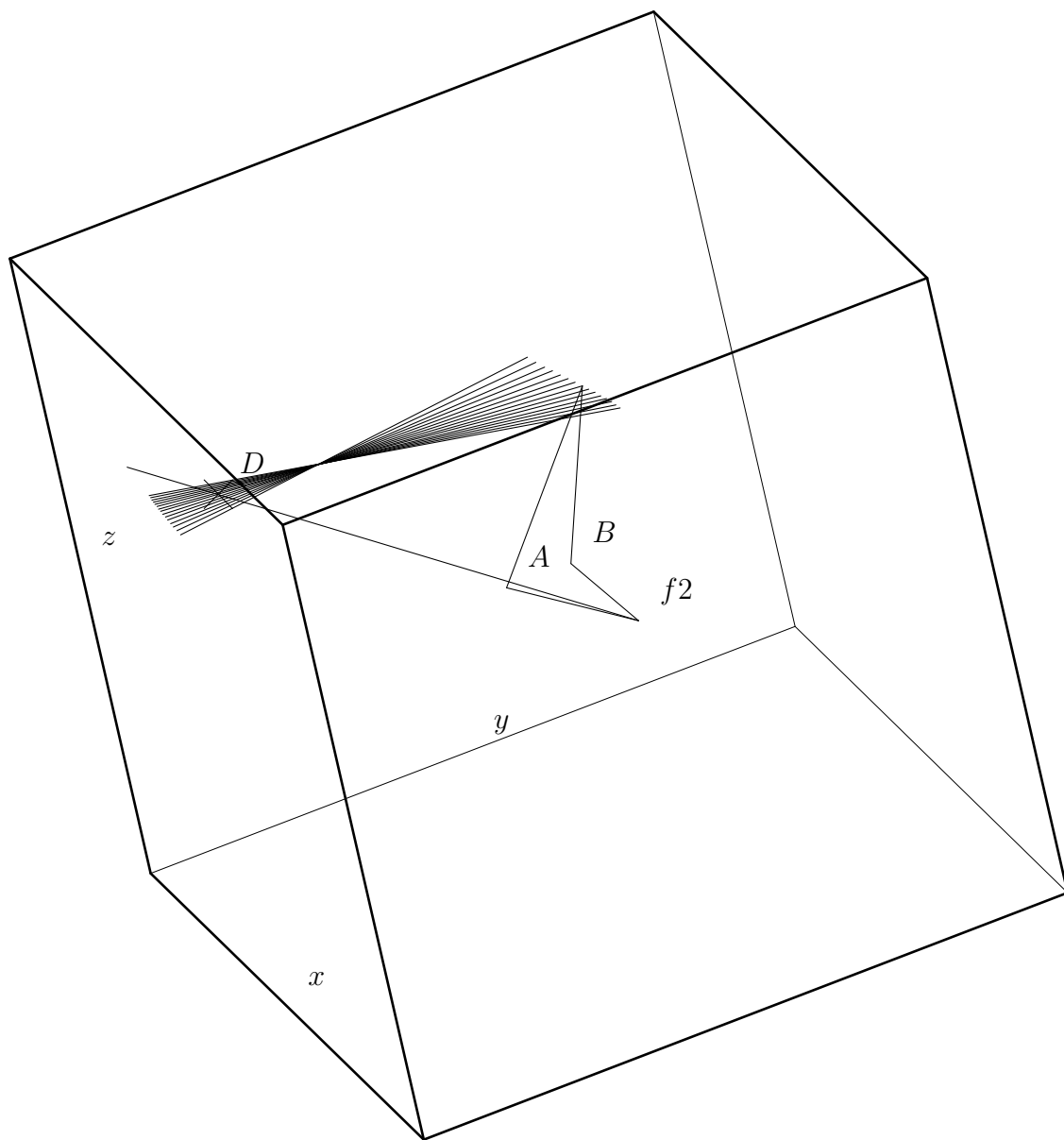
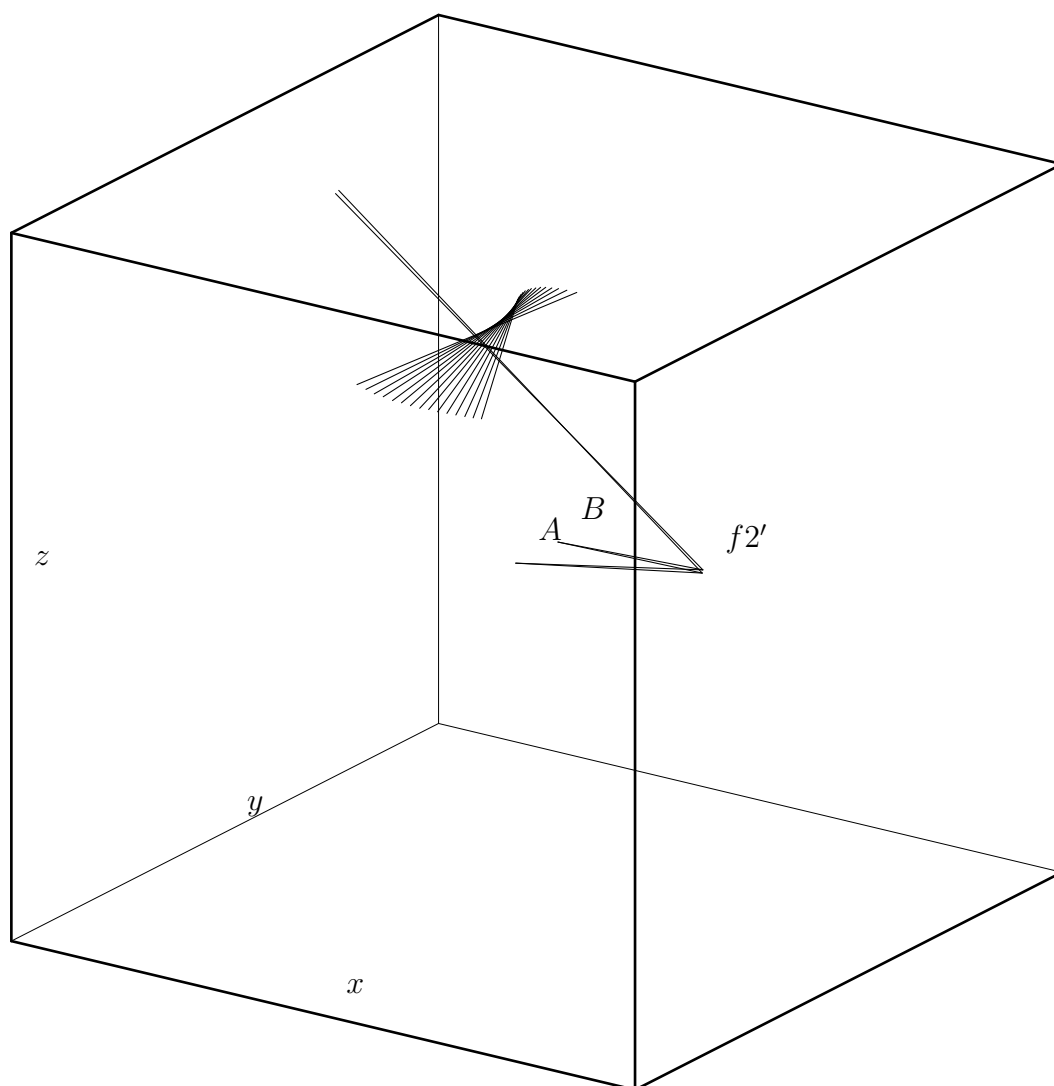


Figure 6: Forth observed point  $D$ . Assume we fix  $f1$ . Candidate surface for  $D$  derived from previous figure.



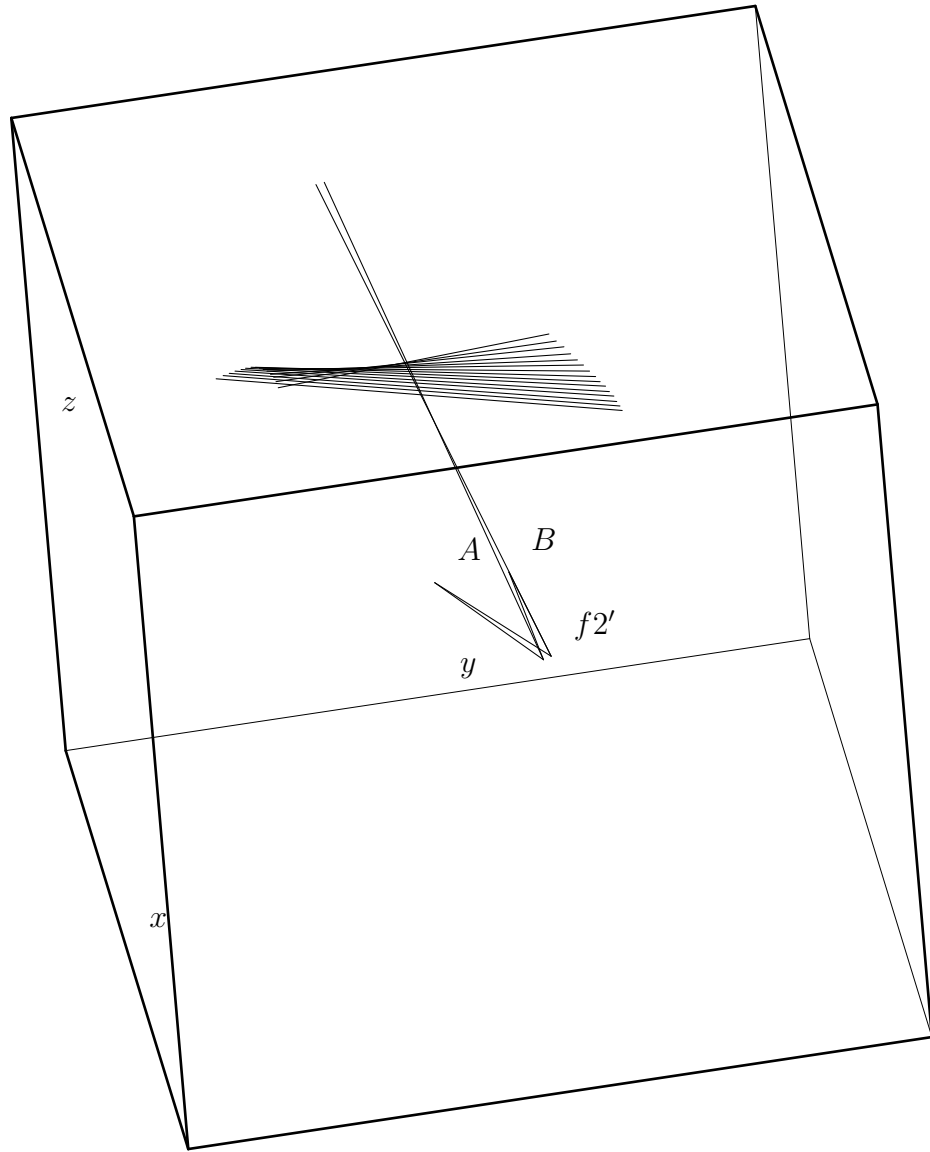
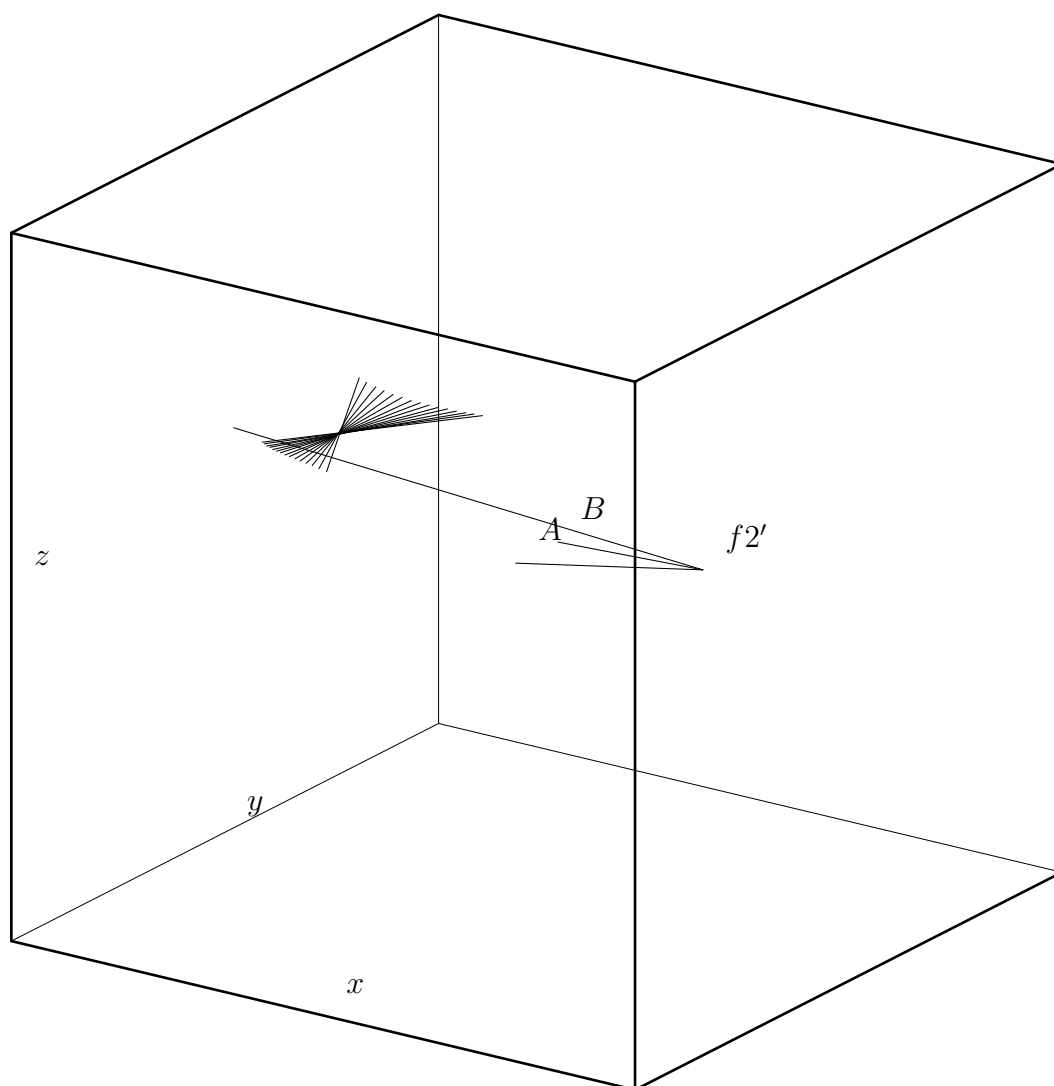


Figure 7: Three observed points: A,B,C.  $f_1'$  slightly rotated in  $z=0$  plane circle with respect to Fig. 5. Surface consisting of lines connecting candidate C points with with corresponding  $f_2'$  locations.



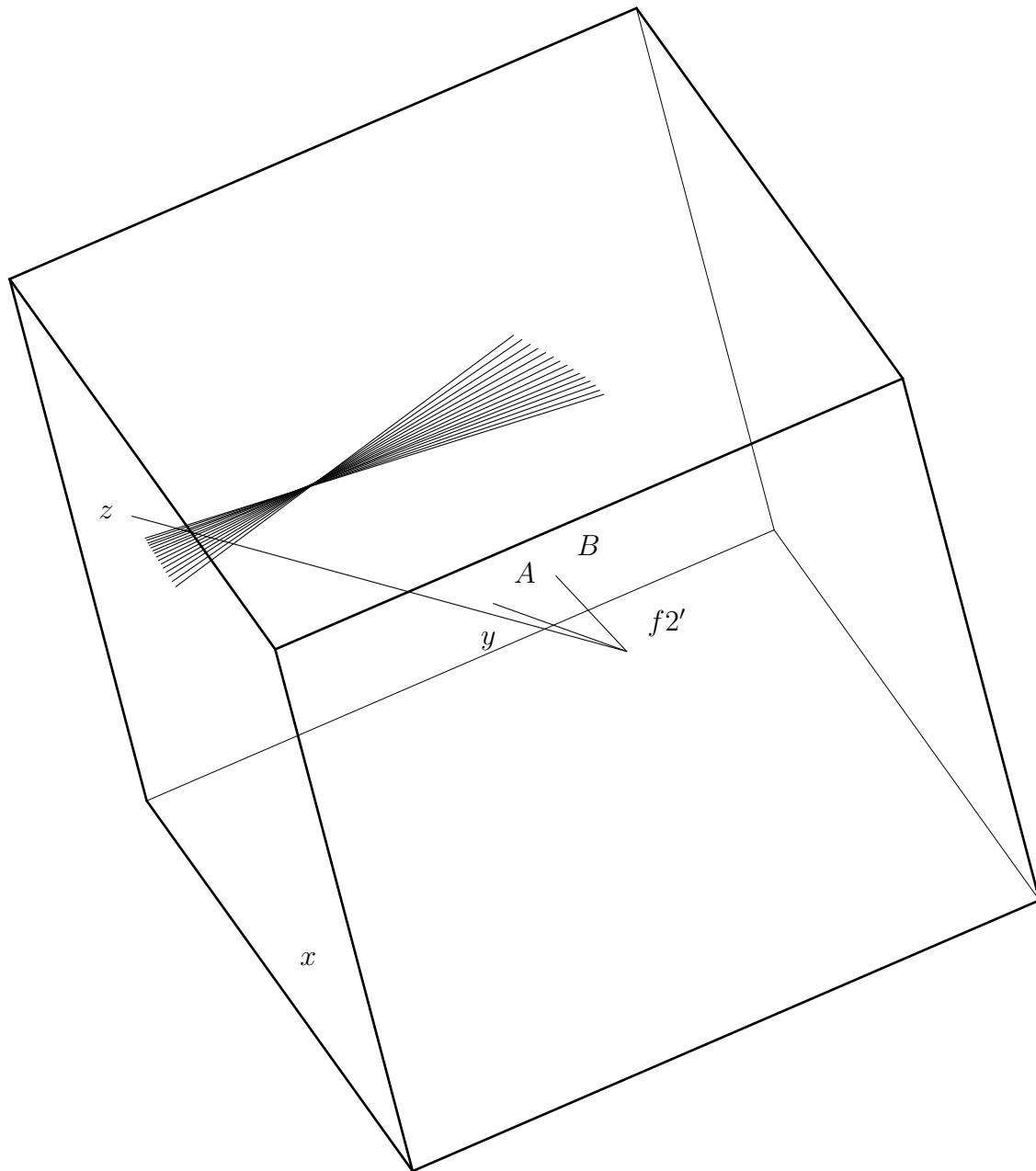


Figure 8: Forth observed point D.  $f1'$  as in previous figure. Candidate surface for D derived from previous figure.

$a'=f_1A'$  ( $=f_1A$ ),  $b'=f_1B'$  ( $=f_1B$ ),  $c'=f_1C'$  ( $=f_1C$ ),  $d'=f_1D'$  ( $=f_1D$ ),  $a''=f_2A''$  ( $=f_2A$ ),  $b''=f_2B''$  ( $=f_2B$ ),  $c''=f_2C''$  ( $=f_2C$ ),  $d''=f_2D''$  ( $=f_2D$ ), assuming at the same time that the camera was moving rather than the four point body itself. (See scene in Fig.1 <sup>2</sup> - REMARK: All the Figures 1-8 are orthogobnal projections of an imaginary scene, where the coordinate system origin is right in the center of the cube drawn in each image. Each of figures 1-8 consists of two images: the first one was taken uniformly from the same projection angle to achieve cross reference among figures. The second image represents the same scene but from a different side, that side which seemed most reasonable for the author of this paper). Obviously, our (fixed) observables are the angles  $Af_1B$ ,  $Af_1C$ ,  $Af_1D$ ,  $Bf_1C$ ,  $Bf_1D$ ,  $Cf_1D$ ,  $Af_2B$ ,  $Af_2C$ ,  $Af_2D$ ,  $Bf_2C$ ,  $Bf_2D$ ,  $Cf_2D$ ,

We can assume (due to the unknown scaling) that we know the length of the edge AB. We fix further the plane p' containing A,B and  $f_1$  (as the  $z=0$  plane in the figures). Within this plane the point  $f_1$  can be any point on a circle, determined by the size of the (fixed) angle  $Af_1B$  and location of two points A,B belonging to it - see Fig.2. (Strictly speaking it lies on one of the two circle fragments cut out by points A and B - this is a well knowwn fact from elementary geometry). Let us select and fix one of those candidate points for  $f_1$ .

Under these circumstances the point  $f_2$  belongs to a rotational surface - with line AB being the rotational axis - containing all the circles determined by the size of the (fixed) angle  $Af_2B$  and location of points A,B belonging to all of them (see Fig.2, for imagination of geometric place of  $f_2$ ). The degrees of freedom for positioning  $f_1$  on this rotational body are indicated in Figs 3 and 4. Fig.3 shows the surface drawn in space by the line  $c''$  if we fix the angle  $f_2AB$  and let rotate  $f_2$  around the AB axis. Fig.4 shows aa fragment of surface drawn in space by the line  $c''$  if we fix an  $f_2AB$  circle and let  $f_2$  move around this circle.

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<sup>2</sup>Figures have been added, and the content of this subsection has been changed to utilize images



Let us concentrate now on identification of points A,B,C in space that is after fixing position of A and B in space we try to find the geometrical place of crossing of straight lines  $c'$  and  $c''$  (Fig.5) when  $f_1$  is fixed and  $f_2$  can move freely (up to geometrical constraints indicated above). If we select an angle  $f_2AB$  and let  $f_2$  rotate around the AB axis then a paraboloidal surface is drawn by straight line  $c''$  as indicated in Fig.3, so that the straight line  $c'$  in general has a chance crossing this surface, just indicating the position of point C in space. Then in general, due to continuity,  $c'$  crosses also in the neighbourhood the surface drawn by  $c''$  for a neighbouring angle  $f_2AB$ . So for a continuous set of angles  $f_2AB$  there exists a (continuous) curve  $r$  such that for any point  $f_2$  lying on it the lines  $c'$  and  $c''$  cross one another. If we let  $f_2$  move along this curve  $r$  then  $c''$  will draw in space a surface indicated in Fig.5.

Now let us consider the possibility of lines  $d'$  and  $d''$  crossing each other when we have fixed the point  $f_1$ . When the point  $f_2$  runs along the curve  $r$ , the straight line  $d''$  draws a surface in space (Fig.6). Let us assume that for the current position of point  $f_1$  the line  $d'$  crosses in fact this surface (Fig.6). Then we have matched the two images into a four point 3-D body.

However, let us investigate neighbouring positions of the point  $f_1$  on its circle in the plane  $p'$  (see Fig.7 and 8) - that is we do not assume fixed position of  $f_1$  anymore. We see with ease that in general case we can meet all the four pairs of lines  $a'$  and  $a''$ ,  $b'$  and  $b''$ ,  $c'$  and  $c''$ ,  $d'$  and  $d''$  also in the vicinity of the original matching - compare Fig.5 with 7, and 6 with 8 - because for a slightly shifted point  $F_1$  the curve  $r$  for geometrical place of  $f_2$  for meeting three line pairs ( $a'-a''$ ,  $b'-b''$ ,  $c'-c''$ ) will move slightly in space, so will the surface drawn by  $c''$  on curve  $r$  (see Fig.7), and so will the surface drawn by  $d''$  (see Fig.8). Therefore, the slightly moved  $d'$  will in general have the chance to hit also the slightly moved surface of  $d''$ .

Hence in fact **there exists an unlimited set of possible "recovered" four point bodies corresponding to given two perspective images.**

If we combine these results with the results of section 2, we see that structure and motion cannot be uniquely determined from the data exploited in <sup>[1]</sup>. The fact of obtaining some experimental results in <sup>[1]</sup> can be explained simply by the nature of approximate algorithms used there in combination with a bit complicated spatial structure of the set of four point bodies corresponding to the given two projection frames.

### 3.3 Fundamental Errors in the Proof Given by Wang

Let us confront the above results with the (length and angle) invariant method applied in the paper <sup>[1]</sup>. Clearly the estimates of the number of degrees of freedom and of information content will be different from those exploited in the previous subsections. However, the balance between them will remain the same. As stated in <sup>[1]</sup>, the four points deliver 6 independent invariants (lengths of line segments joining each pair of points). The number of variables introduced by the invariant method is equal 8 (two for each point, as there are two frames). One unknown may be bound by the non-determinability of the scale of the traced object. So we have 6 equations in 7 independent variables which is obviously unsolvable.

Now let us look at the introduction of the line (whether we consider first the line, then the points or vice versa should not change the estimates of the number of unknowns and the number of equations). Obviously, as stated in <sup>[1]</sup>, 4 new variables are introduced this way. So we would need 5 more independent equations (5 more invariants) to reach a solvable equation system. Given 4 non-coplanar points in 3-D space, we can span a coordinate system within this space. It is easily shown, that at least two triples of points out of those four span planes cut by the straight line. To determine the cutting point within each plane we need two distances - from two of the plane spanning points. Hence, four independent invariants are introduced instead of five which are needed. So the reconstruction process has to fail.

At this point we shall make the remark that the above-mentioned four distances do not determine the position of the line uniquely, as distances from two points in a plane determine not a single, but rather two points lying symmetrically, so we get in fact four straight lines out of the distance invariants mentioned above. To get a unique line, a fifth distance may be required (e.g. of the added line to one of the four points). However, this distance would not be independent of the other ones. The first four distances may be selected out of a continuum (in such a way as to let respective circles cross each other), but the fifth distance may obtain only values from a discrete set of four values. However, we do not need to restrict ourselves to distance-like invariants. In fact, if we bind to the four points a coordinate system spanned by them then the four real-valued coordinates of the straight line within this coordinate system are the invariants uniquely determining the position of the line. Hence only four truly independent equations may be established which contradicts the argument and the formula of Wang given in Section 2.4. on

page 1070<sup>[1]</sup>. (Wang claims there that a line introduces six independent equations as it were uniquely defined by 6 constraints).

## 4 Recovery With Five Traceable Points And Two Prospective Images

It has already been proposed to use 5 traceable points to recover structure and motion parameters from two perspective images<sup>[5]</sup> (compare also Roach and Aggarwal<sup>[6] [7]</sup>). Let us propose here another construction of an equation system doing the job, based on the construction described in section 3.2 of this

paper.

Let us imagine now we have two perspective projections of a five point body. Instead of consid-

ering the points  $A', B', C', D', E'$  and  $A'', B'', C'', D'', E''$  being projections of the five points  $A, B, C, D, E$  in the first and the second projection, respectively and the focal points  $f_1$  and  $f_2$  let us consider the straight lines  $a' = f_1 A' (= f_1 A)$ ,  $b' = f_1 B' (= f_1 B)$ ,  $c' = f_1 C' (= f_1 C)$ ,  $d' = f_1 D' (= f_1 D)$ ,  $e' = f_1 E' (= f_1 E)$ ,  $a'' = f_2 A'' (= f_2 A)$ ,  $b'' = f_2 B'' (= f_2 B)$ ,  $c'' = f_2 C'' (= f_2 C)$ ,  $d'' = f_2 D'' (= f_2 D)$ ,  $e'' = f_2 E'' (= f_2 E)$ , assuming at the same time that the camera was moving rather than the four point body itself.

We can assume (due to the unknown scaling) that we know the length of the edge  $AB$ . We fix further the plane  $p'$  containing  $A, B$  and  $f_1$ . Within this plane the point  $f_1$  lies on a circle (determined by the size of the angle  $Af_1B$ ) containing the points  $A, B$ . Under these circumstances the point  $f_2$  belongs to a rotational surface containing all the circles containing  $A$  and  $B$  and determined by the size of the angle  $Af_2B$ .

The position of the focal point  $f_1$  may be characterized by the angle  $ABf_1$ , the position of the focal point  $f_2$  may be characterized by the angle  $ABf_2$  and by the rotational angle of the  $AB$ -axis (3 variables). We shall require that the straight lines  $c', d', e'$  meet with  $c'', d'', e''$  respectively. Analytically, each of the lines is characterized by two vectors (determined from the variables previously described) and a freely ranging parameter  $(p + t * v)$ , so the six lines introduce 6 variables. They provide us with 9 equations (3 for each dimension of any meeting line pair). So we obtain an equation system with 9 equations in 9 variables.

## 5 On The Recovery Under Orthogonal Projection

It is an interesting question to investigate the possibility of reconstruction of structure and motion from multiframe under orthogonal projection. As mentioned in <sup>[3]</sup>, it is possible to recover them from three traceable points and three images having a quadratic equation system, which may be simplified to a linear one if four images are available. This section will exploit those results to show the possibility of reconstruction from two images when four points are traceable and the possibility of

simplification to linear equation systems whenever either five points are traceable (instead of four) or three images are available (instead of two). The respective subsections will be preceded by a degree of freedom consideration.

## 5.1 Degrees of Freedom for Orthogonal Projection

As in case of perspective projections, each point introduces 3 df in the first frame, each line - 4 df minus one df for the whole body as there exists no possibility of determining the initial depth of the body in the space. The motion introduces for each subsequent frame 5 df only, because the motion in the direction orthogonal to the projection plane has no impact on the image. In general, with  $p$  points and  $s$  straight lines forming the rigid body traced over  $k$  frames we have

$$-1 + 3 * p + 4 * s + 5 * (k - 1)$$

degrees of freedom against

$$k * (2 * p + 2 * s)$$

pieces of information available from  $k$  images.

Thus we shall have the balance

$$-1 + 3 * p + 4 * s + 5 * (k - 1) \leq k * (2 * p + 2 * s) \quad (2)$$

to achieve recoverability.

Let us consider some combinations of parameters:

- for  $k=3$  frames,  $p=3$  points we get

$$-1 + 3 * p + 4 * s + 5 * (k - 1) = 18 = k * (2 * p + 2 * s) = 18$$

- for  $k=2$  frames,  $p=4$  points we get

$$-1 + 3 * p + 4 * s + 5 * (k - 1) = -1 + 12 + 5 = 16 = k * (2 * p + 2 * s) = 2 * 2 * 4 = 16$$

On exploiting<sup>3</sup> straight line component of the above equation see<sup>[10]</sup>, and on non-geometrical balancing degrees of freedom see<sup>[8]</sup>.

## 5.2 Structure and Motion for 4 Point Correspondences

Kłopotek<sup>[8]</sup> presented a method for recovery of structure and motions parameters for 3 traceable points and 3 frames under orthogonal projection. Let us briefly review the results as they form a basis for considerations of this section.

Let A,B,C be the traced points of a rigid body, and  $A_i, B_i, C_i$  their respective projections within the  $i^{th}$  frame. Let  $a, b, c, a_i, b_i, c_i$  denote the lengths of straight line segments  $BC, AC, AB, B_iC_i, A_iC_i, A_iB_i$ , respectively. Then for each frame one of the following relationships holds:

Either

$$\sqrt{a^2 - a_i^2} + \sqrt{b^2 - b_i^2} - \sqrt{c^2 - c_i^2} = 0$$

or

$$\sqrt{a^2 - a_i^2} - \sqrt{b^2 - b_i^2} + \sqrt{c^2 - c_i^2} = 0 \quad (3)$$

or

$$-\sqrt{a^2 - a_i^2} + \sqrt{b^2 - b_i^2} + \sqrt{c^2 - c_i^2} = 0$$

(which is easily seen from geometrical relationships, presented analytically and graphically<sup>4</sup> by Kłopotek<sup>[4]</sup>). So we have three equations, for  $i=1,2$ , and  $3$ , in three unknowns,  $a, b, c$ . As any of the above relationships gives after a twofold squaring:

$$\begin{aligned} & a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + \\ & a_i^4 + b_i^4 + c_i^4 - 2a_i^2b_i^2 - 2a_i^2c_i^2 - 2b_i^2c_i^2 = \\ & = 2a^2(a_i^2 - b_i^2 - c_i^2) + 2b^2(-a_i^2 + b_i^2 - c_i^2) + \end{aligned} \quad (4)$$

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<sup>3</sup>Additional information

<sup>4</sup>Additional information

$$2c^2(-a_i^2 - b_i^2 + c_i^2)$$

which is quadratic in  $a^2, b^2, c^2$ , hence solvable by exploitation of respective methods.

So let us consider a rigid body with four points over two frames (i=1,2).

Let  $A, B, C, D$  be the traced points of a rigid body, and  $A_i, B_i, C_i, D_i$  their respective projections within the  $i^{th}$  frame. Let  $a, b, c, d, e, f, a_i, b_i, c_i, d_i, e_i, f_i$  denote the lengths of straight line segments  $BC, AC, AB, AD, BD, CD, B_iC_i, A_iC_i, A_iB_i, A_iD_i, B_iD_i, C_iD_i$ , respectively. Then for each frame three relationships hold:

$$\sqrt{e^2 - e_i^2} = \sqrt{a^2 - a_i^2} + \sqrt{f^2 - f_i^2}$$

and

$$\sqrt{d^2 - d_i^2} = \sqrt{b^2 - b_i^2} + \sqrt{f^2 - f_i^2} \quad (5)$$

and

$$\sqrt{e^2 - e_i^2} = \sqrt{c^2 - c_i^2} + \sqrt{d^2 - d_i^2}$$

or another proper variation according to the possibilities mentioned for three points.

Please notice that we have also a fourth relationship related to the triangle ABC:

$$\sqrt{a^2 - a_i^2} = \sqrt{b^2 - b_i^2} + \sqrt{c^2 - c_i^2}$$

but we make no use of it as it is linearly dependent on the three previous ones.

In this way we got 6 equations (3 for each of the two frames) in six variables a,b,c,d,e,f. The respective twofold squaring leads to quadratic equations.

### 5.3 Linearization of the Equation System

In the opinion of the author of this paper it is easier to solve a linear equation system than a quadratic one, especially if no good guess values are available to start an iteration with. For this reason he considers it to be a good practice to exploit any available redundancy to make the problem a linear one and only to exploit the redundancy for noise reduction after finding a first satisfying approximation. He is not isolated in this thinking as may be seen from Weng<sup>[2]</sup> where 13 lines were traced (instead of sufficing 6).

Kłopotek<sup>[4]</sup> simplified the equation system (4) for 3 traceable points by using four instead of three frames and subtracting the twofold squared equation for the first frame from those of the other ones. So one obtains three equations of the form for  $i=2,3,4$ :

$$\begin{aligned}
 & a_i^4 + b_i^4 + c_i^4 - 2a_i^2 b_i^2 - 2a_i^2 c_i^2 - 2b_i^2 c_i^2 - \\
 & a_1^4 - b_1^4 - c_1^4 + 2a_1^2 b_1^2 + 2a_1^2 c_1^2 + 2b_1^2 c_1^2 = \\
 & = 2a^2(a_i^2 - b_i^2 - c_i^2 - a_1^2 + b_1^2 + c_1^2) + \\
 & 2b^2(-a_i^2 + b_i^2 - c_i^2 + a_1^2 - b_1^2 + c_1^2) + \\
 & 2c^2(-a_i^2 - b_i^2 + c_i^2 + a_1^2 + b_1^2 - c_1^2)
 \end{aligned} \tag{6}$$

which are linear in  $a^2, b^2, c^2$ , hence solvable by exploitation of respective methods. (No linear dependence is introduced as a new frame is exploited unless the motion has a very special form).

The very same approach may be used with the four points: instead of 2 frames we take one more. Then from (5) we obtain the linear equation system for  $i=2,3$ :

$$\begin{aligned}
 & a_i^4 + e_i^4 + f_i^4 - 2a_i^2 e_i^2 - 2a_i^2 f_i^2 - 2e_i^2 f_i^2 - \\
 & a_1^4 - e_1^4 - f_1^4 + 2a_1^2 e_1^2 + 2a_1^2 f_1^2 + 2e_1^2 f_1^2 = \\
 & = 2a^2(a_i^2 - e_i^2 - f_i^2 - a_1^2 + e_1^2 + f_1^2) +
 \end{aligned}$$



$$2e^2(-a_i^2 + e_i^2 - f_i^2 + a_1^2 - e_1^2 + f_1^2) + \\ 2f^2(-a_i^2 - e_i^2 + f_i^2 + a_1^2 + e_1^2 - f_1^2)$$

and

$$\begin{aligned} & d_i^4 + b_i^4 + f_i^4 - 2d_i^2b_i^2 - 2d_i^2f_i^2 - 2b_i^2f_i^2 - \\ & d_1^4 - b_1^4 - f_1^4 + 2d_1^2b_1^2 + 2d_1^2f_1^2 + 2b_1^2f_1^2 = \\ & = 2d^2(d_i^2 - b_i^2 - f_i^2 - d_1^2 + b_1^2 + f_1^2) + \\ & 2b^2(-d_i^2 + b_i^2 - f_i^2 + d_1^2 - b_1^2 + f_1^2) + \\ & 2f^2(-d_i^2 - b_i^2 + f_i^2 + d_1^2 + b_1^2 - f_1^2) \end{aligned} \quad (7)$$

and

$$\begin{aligned} & d_i^4 + e_i^4 + c_i^4 - 2d_i^2e_i^2 - 2d_i^2c_i^2 - 2e_i^2c_i^2 - \\ & d_1^4 - e_1^4 - c_1^4 + 2d_1^2e_1^2 + 2d_1^2c_1^2 + 2e_1^2c_1^2 = \\ & = 2d^2(d_i^2 - e_i^2 - c_i^2 - d_1^2 + e_1^2 + c_1^2) + \\ & 2e^2(-d_i^2 + e_i^2 - c_i^2 + d_1^2 - e_1^2 + c_1^2) + \\ & 2c^2(-d_i^2 - e_i^2 + c_i^2 + d_1^2 + e_1^2 - c_1^2) \end{aligned}$$

This linear equation system is easily solved.

Another possibility of linearization is to stay with 2 frames but to use 5 traceable points instead of four. With 5 traceable points A,B,C,D,E we are interested in recovering 9 line segments:  $AB$ ,  $AC$ ,  $BC$ ,  $AD$ ,  $BD$ ,  $CD$ ,  $AE$ ,  $BE$ ,  $CE$ . The tenth line segment  $DE$  is also of interest, though available from the other ones. Let us turn to twofold squared equations for each of the 10 triangles  $ABC$ ,  $ABD$ ,  $ACD$ ,  $BCD$ ,  $ABE$ ,  $ACE$ ,  $BCE$ ,  $ADE$ ,  $BDE$ ,  $CDE$ . For the triangle  $ABC$  we have:

$$BC_2^4 + AC_2^4 + AB_2^4 - 2BC_2^2AC_2^2 - 2BC_2^2AB_2^2 - 2AC_2^2AB_2^2 -$$

$$\begin{aligned}
& BC_1^4 - AC_1^4 - AB_1^4 + 2BC_1^2 AC_1^2 + 2BC_1^2 AB_1^2 + 2AC_1^2 AB_1^2 = \\
& = 2BC^2(BC_2^2 - AC_2^2 - AB_2^2 - BC_1^2 + AC_1^2 + AB_1^2) + \\
& \quad 2AC^2(-BC_2^2 + AC_2^2 - AB_2^2 + BC_1^2 - AC_1^2 + AB_1^2) + \\
& \quad 2AB^2(-BC_2^2 - AC_2^2 + AB_2^2 + BC_1^2 + AC_1^2 - AB_1^2)
\end{aligned} \tag{8}$$

and so on for each of the ten triangles. Notice that though the not squared equations for each frame are linearly dependent, the two-fold squared subtracted equations are not linearly dependent. Hence the equation system with ten linear equations in 10 unknown squared line segment lengths  $AB^2$ ,  $AC^2$ ,  $BC^2$ ,  $AD^2$ ,  $BD^2$ ,  $CD^2$ ,  $AE^2$ ,  $BE^2$ ,  $CE^2$ ,  $DE^2$  is solvable.

## 6 Discussion and Conclusions

This paper was dealing with the problem of recovery of 3-D structure and motion from multiframe. In the past, this problem has been discussed under various geometrical and physical settings. Roach and Aggarwal <sup>[6]</sup> <sup>[7]</sup> researched on motion and structure recovery tracing points under perspective projection assuming static scene and moving camera. They showed that five points in two views are needed to recover the structure and motion parameters. Their solution involved a system of 18 highly nonlinear equations. Nagel <sup>[5]</sup> proposed a simplified equation system by separating solution for the translation vector and the rotation matrix, with rotation matrix being determined by a system of 3 equations in three motion parameters. Weng et al <sup>[2]</sup> recovered structure and motion from line correspondences only using three frames and 13 lines, obtaining a linear equation system solving the problem.

Simpler solutions of the problem of recovery of structure and motion from multiframe are achievable under orthogonal projections and under special assumptions concerning the physical motion, which allow for reduction of the quantity and quality of required traceable features and/or the complexity of the solution. Lee <sup>[9]</sup> recovered structure and motion tracing two points only under orthog-

onal projection from four frames assuming constant rotational speed of the traced body. Kłopotek<sup>[8]</sup> also dealt with the case of two traceable points, assuming free fall of the body in homogeneous force field.

For other related papers see [10]-[17].

The paper of Wang<sup>[1]</sup> represents an attempt to press down further the quantity and quality of required traceable features under perspective projection. Wang attempts to use only four points and a line (instead of the earlier achievement of five points) over two frames. A line introduces generally more degrees of freedom (4 against 3 introduced by a point) and its position may be traced more reliably than that of a point<sup>[2]</sup>. The gain would be obvious: the less features are required as a minimum the more stability may be pressed out of the actually available redundancy.

However, the current paper points at a basic weakness of the Wang's experimentally widely elaborated paper: the claim of the possibility of the recovery of structure and motion from the claimed minimal set of features does not hold. The following arguments were presented: a) a line does not contribute anything to recovery from two frames only, b) four points alone do not suffice to recover structure and motion. This has been shown by a) evaluating the number of degrees of freedom against the amount of information obtainable from the frames, b) by constructing an infinite set of candidate rigid bodies giving two predefined projections.

We have also pointed at the errors in the paper of Wang which led to those incorrect results. Three claims of Wang are wrong. The first error is a cosmetic one: Wang claims that a (sufficiently big) set of distance and/or angular constraints determines uniquely the shape of a rigid body consisting of points and lines. In reality it is not a full uniqueness but a uniqueness up to a mirror symmetry. The basic error is a serious one: Wang claims that a constraint selecting a value out of a

countable finite set of values is independent of those constraints which led to such a restriction of the set of values. The second wrong claim is in some sense a consequence of the first one: if symmetries could be fully eliminated by adding constraints then these constraints were independent. The third error just summarizes the previous ones: Wang missed the possibility of defining invariants in such a way as to eliminate ambiguities resulting from symmetries: he should have substituted distances by coordinates in a coordinate system bound to the recovered rigid body.

This paper shows also that under orthogonal projection the four points would suffice with two frames to recover structure and motion. Furthermore, if one more point were available or the body were traced over one more frame then a linear equation system may be constructed to find the structure and motion parameters.

Last not least I want to apologize to the Authors of the paper <sup>[1]</sup>. The critical remarks made here should not undermine the value of the labour behind the paper nor the value of large portions of the results published because much of the simulation and experimental work has been done with a number of traceable points sufficient for recovery purposes. Hence practical conclusions and hints from that research work remain valid to a large extent.

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**Summary:** Wang et al.<sup>[1]</sup> claimed a new method of recovering structure and motion parameters from a sequence of two frames (under perspective projection) applicable whenever correspondences for four points and a straight line belonging to a rigid body over two frames may be established..In this paper we would like to deny the results of Wang<sup>[1]</sup> raising two fundamental claims:

- A line does not contribute anything to recognition of motion parameters from two images because:
  - from two images we obtain only as much information as to position a line relatively to other, earlier recovered objects,
  - three projections of a line are necessary to extract from them some contribution to recover structure and motion parameters of a rigid body.
- Four traceable points are not sufficient to recover motion parameters from two perspective projections, because:
  - the number of degrees of freedom connected with the four point rigid body and the motion between the frames exceeds the amount of information that can be extracted from a pair of images,
  - a pair of images may correspond to an infinite number of rigid four point bodies

We point at basic errors in the theoretical part of Wang's paper and try to explain briefly the validity of experimental results.

We show also that four traceable points are sufficient to recover motion parameters from two frames under orthogonal projection and that five traceable points in two frames or four traceable points in three frames are sufficient to simplify the solution of the reconstruction problem under orthogonal projection to solving a linear equation system.

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